Indian Statistical Institute, Bangalore
B. Math. First Year
First Semester - Analysis I

Mid-Semester Exam Date : Sept 08, 2014

This paper carries 25 marks. Maximum Marks you can get is 20.

1. Let $f : [1, \infty) \to [0, \infty)$ be any continuous decreasing function. Then show that
$$\sum_{1}^{\infty} f(j) \text{is summable} \Leftrightarrow \int_{1}^{\infty} f(u)du < \infty.$$
[3]

2. Let $a_1, a_2, \ldots$ be any sequence of complex numbers. If $\sum_{1}^{\infty} |a_n|$ is summable, then $\sum_{1}^{\infty} a_n$ is summable. [2]

3. Let $\{x_1, x_2, \ldots\}$ be a sequence of nonzero complex numbers. Assume that
$$\limsup_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = L < 1$$
Then $\sum_{1}^{\infty} |x_n| < \infty$ [3]

4. The series $\sum_{1}^{\infty} a_n b_n$ is summable if $A_n = a_1 + a_2 + \ldots + a_n$ is a bounded sequence and the sequence b_n decreases to 0. [4]

5. Let w_1, w_2, w_3 be the roots of $x^3 - 1 = 0$. Define $a_n = w_3$ if n is divisible 3, $a_n = w_2$ if $n \equiv 2 \mod 3$ and $a_n = w_1$ if $n \equiv 1 \mod 3$. Show that $\sum_{1}^{\infty} \frac{a_n}{\log(n+100)}$ is summable. [3]

6. Let $a_1, a_2, a_3, \ldots$ be a real sequence bounded below. Let $\alpha = \liminf_{j\to\infty} a_j$. Then for each $\delta > 0$, show that there exists k_0 such that $a_k \geq \alpha - \delta$ for all $k \geq k_0$. [1]

   Discuss the summability of the following examples.

7. $\sum_{n=1}^{\infty} p^n n^p \qquad p > 0$ [3]

8. $\sum_{100}^{\infty} \frac{1}{n \log n (\log \log n)^p} \qquad p > 0$ [3]

9. $\sum_{100}^{\infty} n^p \left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}\right)$ [3]